Fenchel Duality with Auxiliary Parameters

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2007

Outline

Functions with Auxiliary Parameters

We will frequently need to deal with functions of the form

$$f'(y) = \inf_{u} f(y, u)$$

- $f(y, u)$ being convex in y for fixed u is *not* a sufficient condition for f' to be convex.
- If f is convex in y and u simultaneously then f' is convex.
- Throughout today's lecture, we define f' as above, and assume that $y \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

Auxiliary Fenchel Conjugate

Theorem

$$f^*(z,0) = f'^*(z)$$

$$\begin{aligned} f^*(z,0) &= \sup_{y,u}\{y^t z + u^t 0 - f(y,u)\} \\ &= \sup_{y,u}\{y^t z - f(y,u)\} \\ &= \sup_{y}\{y^t z - \inf_u f(y,u)\} \\ &= f'^*(z) \end{aligned}$$

If you remember one thing about Fenchel duality for functions with auxiliary parameters, remember this.

Subgradient Relations for Functions with Auxiliaries

Theorem

$$z \in \partial f'(y) \text{ and } f'(y) = f(y, u) \Leftrightarrow (z, 0) \in \partial f(y, u).$$

Proof: (see next section)

Outline

Functions with Auxiliary Parameters

We will frequently need to deal with functions of the form

$$f'(y) = \inf_u f(y, u)$$

If the infimum is attained for all y where $f'(y)$ is finite, then we say f' is *exact*. If f is ccp then f' is convex and dom $h' \neq \emptyset$, however, this does not guarantee that h' is exact, closed, or that dom $(-h') = \emptyset$.

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The Achievable W.

- Define $W = \{w \in \mathbb{R}^m : \exists z \in \mathbb{R}^n, (z, w) \in \text{dom } f^*\}$.
- If $0 \in W$, then, $\forall y, f'(y) > -\infty$. Proof: Suppose $\exists \hat{y}$ s.t. $f'(\hat{y}) = -\infty$. Then $\forall z, f'^*(z) = f^*(z, 0) = \sup_y \{y^t z - f'(y)\} = \infty$, so $0 \notin W$.
- We see that $0 \in W$ is a **sufficient** condition for f' to avoid the value $-\infty$.
- Two easy thought exercises: $f(y, u) = y + u, f(y, u) = y + u^2$.
- Is $0 \in W$ a **necessary** condition for f' to avoid the value $-\infty$? No.

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Subgradient Relations for Functions with Auxiliaries

Theorem

$$z \in \partial f'(y) \text{ and } f'(y) = f(y,u) \Leftrightarrow (z,0) \in \partial f(y,u).$$

$$\begin{aligned} & z \in \partial f'(y) \text{ and } f'(y) = f(y,u) \\ \Rightarrow \quad & f(y,u) - y^t z + f^*(z,0) = 0 \\ \Rightarrow \quad & (z,0) \in \partial f(y,u) \end{aligned}$$

$$\begin{aligned} & 0 = f(y,u) - y^t z + f^*(z,0) \\ \Rightarrow \quad & 0 \geq f'(y) - y^t z + f'^*(z) \\ \Rightarrow \quad & 0 = f'(y) - y^t z + f'^*(z) \text{ and } f'(y) = f(y,u) \end{aligned}$$

In the second derivation, we used $h(y,u) \geq h'(y)$ and Fenchel-Young.

An Exactness Condition

Theorem

If $0 \in int(W)$ then h' is ccp and exact.

For a proof, see “Value Regularization and Fenchel Duality,” Rifkin and Lippert.