Solutions to Fenchel-Legendre Conjugate Exercises

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1. Let $g(y) = f(y) - a^t y$. Derive $g^*(z)$ (in terms of $f^*(z)$).

$$\begin{aligned} g^*(z) &= \sup_y \{y^t z - g(y)\} \\ &= \sup_y \{y^t z - f(y) + a^t y\} \\ &= \sup_y \{y^t (z + a) - f(y)\} \\ &= f^*(z + a). \end{aligned}$$

2. Let $g(y) = af(y)$ (assume $a > 0$). Derive $g^*(z)$.

$$\begin{aligned} g^*(z) &= \sup_y \{y^t z - af(y)\} \\ &= \sup_y \{y^t az/a - af(y)\} \\ &= a \sup_y \{y^t z/a - f(y)\} \\ &= af^*(z/a)\} \end{aligned}$$

3. Let $g(y) = f(a - y)$. Derive $g^*(z)$.

$$\begin{aligned} g^*(z) &= \sup_y \{y^t z - f(a - y)\} \\ &= \sup_y \{(a - y)^t z - f(y)\} \\ &= \sup_y \{(y - a)^t (-z) - f(y)\} \\ &= \sup_y \{y^t (-z) - f(y)\} + a^t z \\ &= f^*(-z) + a^t z. \end{aligned}$$

4. Let $f(y) = \frac{1}{2}(a - y)^2$. Derive $f^*(z)$ two different ways: first directly using the definition of the Fenchel-Legendre conjugate, and then by applying previously known identities.

   Direct:

$$f^*(z) = \sup_y \{yz - \frac{1}{2}(a - y)^2\}$$

   For a given z, differentiation shows that the sup is attained when

$$y = a + z,$$

so

$$\begin{aligned} f^*(z) &= (a+z)z - \frac{1}{2}z^2 \\ &= az + z^2 - \frac{1}{2}z^2 \\ &= \frac{1}{2}z^2 + az \end{aligned}$$

Previously known identities: Defining $g(y) = \frac{1}{2}(-y)^2 = \frac{1}{2}y^2$, the last exercise demonstrated that $f^*(z) = g^*(-z) + az$. We already knew that $g^*(-z) = g^*(z) = \frac{1}{2}z^2$, so we are done. A primary motivation for the Fenchel approach to learning is that simplifications such as this are often possible.

5. Let $f(y) = |y|$. Derive $f^*(z)$.

$$f^*(z) = \sup_y \{yz - |y|\}.$$

If $z = 0$, we see that $f^*(0) = 0$. A similar analysis shows that $f^*(z) = 0$ whenever $|z| \leq 1$.

Suppose $z = 2$. By choosing y arbitrarily large and positive, we can make $y^t z - |y|$ as large as we want, so $f^*(2) = \infty$. A similar analysis shows that $f^*(z) = \infty$ whenever $|z| > 1$.

The graphical method paints the same picture: for $|z| \leq 1$, a hyperplane with slope z that supports $|y|$ passes through the origin, while for $|z| > 1$, there is no hyperplane with slope z that supports $|y|$.